We shall now develop the mathematical model.[1]

The total income of the social security system at time t (for $t \geq 0$) is given by

$$\int_0^g w(t,x)D(t-x)r(t)dx. \tag{3.1}$$

The lump-sum payment due the cohort that retires at time t is (for $t \geq g$) given by

$$\int_0^g w(t-g+x,x)D(t-g)r(t-g+x)e^{\alpha(g-x)}dx. \tag{3.2}$$

We assume (3.1) and (3.2) are equal for each $t \geq g$. Thus,

$$r(t) = \frac{D(t-g)\int_0^g w(t-g+x,x)r(t-g+x)e^{\alpha(g-x)}dx}{\int_0^g w(t,x)D(t-x)dx}. \tag{3.3}$$

From (3.) it is apparent that the tax $r(t)$ is defined in terms of the tax rates $r(t-k)$ where $0 < k \leq g$—in other words the tax rate at the instant of retirement depends on all the tax rates prevailing during the working lifetime of the retiring cohort.

It is also obvious, if we have a given function $r(k)$, for $0 \leq k < g$, that equation (3.3) completely defines $r(t)$ for all $t \geq g$. The function $r(k)$, where $0 \leq k < g$, represents in our model an arbitrary set of starting conditions for the system. We only restrict this function to being positive and continuous. Now given an arbitrary $r(k)$ $(0 \leq k \leq g)$ we might seek a function $r(t)$ that solves (3.3) for all $t \geq g$. This function would be the unique schedule of tax rates that satisfies our requirements (equity, solvency, and administrative simplicity), and is consistent with the given initial conditions. But this solution or tax rate schedule would reflect the influence of the arbitrary initial conditions, in which we have no immediate interest.[2] We are more interested in deducing common characteristics of all tax rate schedules $r(t)$ $(g \leq t \leq \infty)$ that derive from (3.3) and some arbitrary set of initial conditions. These characteristics would presumably reflect the intrinsic influence of the equity, solvency, and administrative simplicity restrictions of the system.

To deduce these common characteristics we make specific assumptions about the form of the functions $w(t,r)$ and $D(t)$. These assumptions are adopted because they are mathematically tractable and because they have been widely used in models dealing with economic growth.[3]

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[1] The model we develop has considerable (an inevitable) common ground with various consumption loans models that have appeared in the economics literature. See, e.g., David Cass and Menahem E. Yaari, "A Re-examination of the Pure Consumption Loans Model," the Journal of Political Economy, vol. LXXIV, No. 4, August 1966, pp. 353–367.

[2] In the appendix to this section we deal briefly (and somewhat elliptically) with the relation of initial conditions to the resulting function $r(t)$ $g \leq t \leq \infty$.

[3] See Edmund S. Phelps, "Golden Rules of Economic Growth" New York, W. W. Norton, 1966.