This may be rewritten as follows:

$$r(t)=\frac{D_0e^{\pi(t-g)}e^{\gamma t}\int_0^g w(x)e^{\gamma(x-g)}r(t-g+x)e^{\alpha(g-x)}dx}{D_0e^{\gamma t}e^{\pi t}\int_0^g w(x)e^{-\pi x}dx}$$

$$r(t)=\frac{e^{g(\alpha-\gamma-\pi)}\int_0^g w(x)e^{-(\alpha-\gamma)x}r(t-g+x)dx}{\int_0^g w(x)e^{-\pi x}dx}. \tag{3.8}$$

After defining $K=\alpha-\gamma$ we rewrite and rearrange (3.8):

$$0=r(t)\int_0^g w(x)e^{-\pi x}dx-e^{g(K-\pi)}\int_0^g w(x)e^{-Kx}r(t-g+x)dx$$

$$0=\int_0^g w(x)e^{-Kx}\{r(t)e^{(K-\pi)x}-e^{(K-\pi)g}r(t-g+x)\}dx. \tag{3.9}$$

We have a solution if the term in braces in (3.9) is zero for every x. This implies

$$r(t)e^{(K-\pi)x}=e^{(K-\pi)g}r(t-g+\alpha)$$
$$r(t)=e^{(K-\pi)(g-x)}r(t-g+x).$$

Thus

$$r(t)=R_0e^{(K-\pi)t} \tag{3.10}$$

is a function that solves (3.9), (3.7), and (3.3) (with appropriate initial conditions).

More significantly, it can be shown that any set of initial conditions will determine a tax rate schedule $r(t)$ that satisfies the following inequalities:

$$R_Le^{(K-\pi)t}\leq r(t)\leq R_Ue^{(K-\pi)t} \tag{3.11}$$

where

$$R_L\leq R_U.$$

This result has applied implications. Recall that $K=\alpha-\gamma$, so that $K-\pi=\alpha-\gamma-\pi$. If $\alpha>\gamma+\pi$ then $K-\pi>0$. If this inequality holds then $R_Le^{(K-\pi)t}$ will increase exponentially and eventually surpass any finite upper limit. Since $r(t)\geq R_Le^{(K-\pi)t}$ $\alpha>\gamma+\pi$ implies that $r(t)$ will increase without bound. Hence, $\alpha>\gamma+\pi$ is inconsistent with the stability constraint.

By a symmetric argument it is apparent that if $\alpha<\gamma+\pi$ then $K-\pi<0$ and $R_Ue^{(K-\pi)t}$ will asymptotically approach zero. Since $r(t)\leq R_U{}^{(K-\pi)t}$, this implies that collection rates and benefits will eventually become trivially small. While this might be consistent with absolutely increasing benefits, it remains probable that $\alpha<\pi+\gamma$ is inconsistent with the social adequacy requirement. Thus our model strongly suggests that the only appropriate longrun rate of interest is

$$\alpha=\gamma+\pi. \tag{3.12}$$